An Introduction to the Fourier Transform

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Basic Measure Theory

- Essentially, a *measure space* (X, μ) is a set X, together with a *measure* μ on X which assigns some subsets of X to values in $[0, \infty]$.
- There can be many different measures on a single set.
- The canonical measure on $\mathbb{R}^n$ is called *Lebesgue measure*, and is denoted m.
- We say a property $P(x)$ holds *almost everywhere (a.e.)* if the set $\{x \in X : \neg P(x)\}$ has measure 0.
- A function $f : X \to \mathbb{R}$ is called *measurable* if it satisfies some basic minimal requirements. All functions will be assumed to be measurable.

Lebesgue Integration

- For a measure space (X, μ) and a nonnegative measurable function $f\colon X \to \mathbb{R}$, we can calculate the *Lebesgue integral* of f, written $\int_X f\, d\mu$. This is a value in $[0, \infty]$.
- We say f is *(Lebesgue) integrable* if $\int_X f\, d\mu < \infty$.
- If $f\colon \mathbb{R} \to \mathbb{R}$ is Riemann integrable, then it is also Lebesgue integrable, and the values of the two integrals coincide.
- For $p \in [1, \infty)$, we define

$$L^p(X) := \left\{ f\colon X \to \mathbb{R} \text{ measurable} \;\middle|\; \left(\int_X |f|^p \, d\mu \right)^{1/p} < \infty \right\} \Big/ \sim$$

where $f \sim g \iff f = g$ a.e.

Key Theorems

- **Theorem:** (*The Dominated Convergence Theorem*) Suppose (X, μ) is a measure space, and for each $n \in \mathbb{Z}^+$, $f_n \colon X \to \mathbb{R}$ is measurable. If there exists a non-negative integrable function $g \colon X \to \mathbb{R}$ such that $|f_n| \leq g$ for all $n \in \mathbb{Z}^+$, then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \lim_{n\to\infty} f_n \, d\mu.$$

- **Theorem:** (*Fubini's Theorem*) Suppose (X, μ) and (Y, ν) are measure spaces, and $f \colon X \times Y \to \mathbb{R}$. If certain conditions are satisfied, then

$$\int_Y \left(\int_X f(x, y) \, d\mu(x) \right) d\nu(y) = \int_X \left(\int_Y f(x, y) \, d\nu(y) \right) d\mu(x).$$

Convolution

- For $f, g \in L^2(\mathbb{R}^n)$, the *convolution* of f and g is the function $f * g$ given by
$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\, g(x - y)\, dy.$$
- It can be shown (using a substitution) that $f * g = g * f$.

Distributions

- For $\Omega \subseteq \mathbb{R}^n$ open, we define

$$\mathcal{D}(\Omega) := \{f \in C^\infty(\mathbb{R}^n) \mid \text{supp}\, f \subseteq \Omega \text{ compact}\}$$

  Elements of this space are called *test functions*.

- The set of all continuous linear functionals from $\mathcal{D}(\Omega) \to \mathbb{R}$ is denoted $\mathcal{D}'(\Omega)$. Elements of this set are called *distributions*.

- The set of *rapidly decreasing functions* on $\mathbb{R}^n$, denoted $\mathcal{S}(\mathbb{R}^n)$, is the set of all functions $f \in C^\infty(\mathbb{R}^n)$ such that for all $m \in \mathbb{Z}^+$ and all multi-indices k, $D^k f < |x|^{-m}$ for sufficiently large $|x|$.

- The set of all continuous linear functionals from $\mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ is denoted $\mathcal{S}'(\mathbb{R}^n)$. Elements of this set are called *tempered distributions*.

- We have $\mathcal{S}(\mathbb{R}^n) \supseteq \mathcal{D}(\mathbb{R}^n)$, and so $\mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$.

The Fourier Transform

- For $\varphi \in \mathcal{S}(\mathbb{R}^n)$, the *Fourier transform* of φ is the function $F[\varphi]$ given by

$$F[\varphi](z) = \int_{\mathbb{R}^n} \varphi(x) e^{iz\cdot x} dx;$$

and the *inverse Fourier transform* of φ is the function $F^{-1}[\varphi](z)$ given by:

$$F^{-1}[\varphi(z)](x) := \frac{1}{(2\pi)^n} F[\varphi(-z)](x).$$

The Fourier Transform: Key Theorems

- **Theorem:** (*Fourier Inversion Theorem*) For $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

  $$F^{-1}[F[\varphi]] = F[F^{-1}[\varphi]] = \varphi.$$

  Proof: Uses the Dominated Convergence Theorem and Fubini's Theorem.

- **Theorem:** (*The Convolution Theorem*) If $f, g \in L^1(\mathbb{R}^n)$, then $f * g \in L^1(\mathbb{R}^n)$, and

  $$F[f * g](z) = F[f](z) \cdot F[g](z).$$

  Proof: Uses Fubini's Theorem.

Sources

This project was supervised by Professor Rasul Shafikov of Western University.

The following sources were consulted in the creation of this project:

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