DIHEDRAL GROUPS, SYMMETRIC GROUPS, AND THE SYMMETRIES OF POLYHEDRA

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ABSTRACT. We were regurgitating group theory in an Elementary Number Theory course and one of us asked

"Is every finite group cyclic?"

Instantly I thought about the dihedral group because of my love for geometry. Another one of them came up with the counterexample of a symmetric group. After learning math together, I came up these questions:

Is $D_4 \leqslant S_4$? Is the symmetry of cube, $S_C \leqslant S_8$? Is $D_8 \leqslant S_C$?

To answer this question, let's flip the sandglass and chase the butterfly.

1. PRELIMINARIES

1.1. **Group.** A group $(G, *)$ is a set G associated with a binary operation $*$ satisfying

(1) For every $a, b, c \in G$, $(a * b) * c = a * (b * c)$
(2) There exists an identity $e \in G$ such that for every $a \in G$, $e * a = a = a * e$
(3) For every $a \in G$ there exists a unique inverse $a^{-1} \in G$ such that $a^{-1} * a = e = a * a^{-1}$

1.1.1. *Order.* The order of a group G is defined as the number of elements in G. We write $o(G) = |G|$. Similarly, the order of an element $g \in G$ is defined as the *least* number n such that $g^n = e$ and we write $o(g) = n$.

1.1.2. *Subgroups.* Let G be a group and $H \subseteq G$. H is a *subgroup* of G if H is also a group with same identity $e \in H \subseteq G$. We write $H \leqslant G$.

1.1.3. *Generators.* Let G be a group and fix $g \in G$. The *cyclic* subgroup *generated* by g is $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$.

1.1.4. *Cyclic Group.* Let G be a group. G is *cyclic* if $\langle g \rangle = G$.

1.1.5. *Generating Set.* Let $g_1, \ldots, g_n \in G$. Then the subgroup generated by the set $S = \{g_1, \ldots, g_n\}$ is denoted as $\langle g_1, \ldots, g_n \rangle = \bigcap_{\substack{H \leqslant G \\ S \subseteq H}} H$. This is the smallest subgroup of G that contains all the finite products of $g_1, \ldots, g_n$.

1.2. **Isomorphism.** An isomorphism is a bijective *homomorphism* $\phi : X \to Y$. A homomorphism is a structure preserving map $\phi : X \to Y$.

1.2.1. *Automorphism.* Let $\phi : X \to X$ be an isomorphism on set X. Then ϕ is an *automorphism* on X and we write $\phi \in \mathrm{Aut}(X)$. $\mathrm{Aut}(X)$ is a group action on X.

1.2.2. *Group Isomorphism.* Let G, H be groups. Then $\phi : G \to H$ is a *group isomorphism* if it is invertible and for every $g, g' \in G$, $\phi(gg') = \phi(g)\phi(g')$.

1.3. **Graph.** A graph $G = (V, E)$ consists of two sets, called the *vertices* and the *edges*, such that $V = \{v_1, \dots, v_n\}$ and $E = \{\{v_i, v_j\} : 1 \leq i \neq j \leq n\} \subseteq V \times V/\sim$ where $\sim: \{v_i, v_j\} \sim \{v_j, v_i\}$ is an equivalence relation.

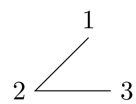

FIGURE 1. Graph with $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{2, 3\}\}$

1.3.1. *Graph Isomorphism.* Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be two graphs. Then $f : V_1 \to V_2$ is a *graph isomorphism* if f is invertible and $\{v_i, v_j\} \in E_1$ is equivalent to $\{f(v_i), f(v_j)\} \in E_2$. The graph isomorphism is an "edge-preserving" bijection.

1.4. **Polygon.** A polygon P_n is a graph (V_n, E_n) with an enclosed area F, called the *face.* The two sets in the graph, called the *vertices* and the *edges* are $V_n = \{v_1, \dots, v_n\}$ and $E_n = \{\{v_i, v_{(i \mod n)+1}\} : 1 \leq i \leq n\}$. F is enclosed by (V_n, E_n).

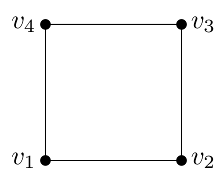

FIGURE 2. Square with $n = 4$

Remark. This definition allows us to categorize the polygon with respect to the number of vertices. It's also possible to categorize it with sum of the angles and their relatives.

There are different types of polygon, the "good" ones and the "bad" ones. A *regular* polygon has equal side length. A *convex* polygon has all angles with degree less than 180°. A *nonconvex* polygon has at least one angle with degree greater than 180°.

We can define a geometrical *invariant* for the polygons. For the convex polygons, the *sum of interior angles* of a convex polygon with n edges, an n-gon, always equals to $(n - 2) \cdot 180°$. The *sum of supplementary angles* of an n-gon always equals to 360°, or 2π, the radian measure of a plane.

Unfortunately, both of these two fail to hold for non-convex polygons. The sum of interior angles could be ranging from 180° (a straight line L) to $(n - 3) \cdot 180°$ (a polygon P_{n-1}). The sum of supplementary angles is even worse; it's undefined as we now introduced the obtuse angles and the reflex angles. Hence we introduce a new measure, the *sum of explementary angles*, which always equals to $(n+2) \cdot 180°$.

Proof. It's easy to show the sum of explementary angles of a convex n-gon P_n satisfies the invariant. Since every vertex on P_n entails an angle of 360°, the sum of the explementary angles of P_n equals to

$$n \cdot 360° - (n - 2) \cdot 180° = (n + 2) \cdot 180°$$

For any non-convex polygons P_n with $n = 2k$, we could always deform the shape into k line segments with one common vertex.

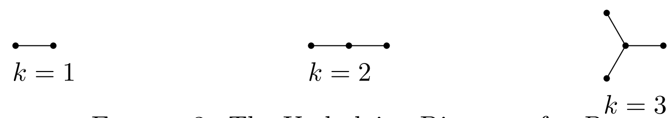

FIGURE 3. The Underlying Diagrams for P_{2k}

When $n = 2k + 1$, we could decompose the polygon P_n into a convex triangle and a non-convex polygon P_{n-3}.

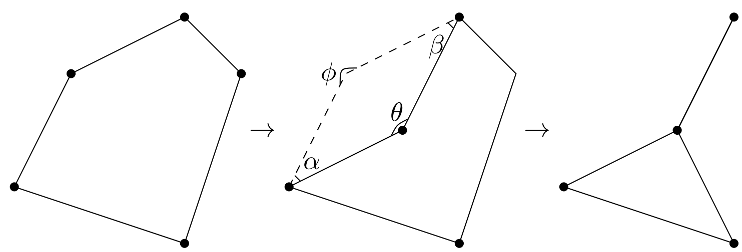

FIGURE 4. Parallelogram Rule for the Sum of Angles

Since $\phi = \alpha + \beta + \theta$, we could convert the explementary angles of the convex vertices into the central node. This gives us the sum of the explementary angles equals to $(n + 2) \cdot 180°$. □

1.5. **Symmetries of Regular Polygon.** Let P_n denote a regular convex polygon with n edges. A symmetry on P_n is a rigid motion that preserves the geometrical structure of P_n. We use r to denote the counterclockwise rotation of $\frac{2\pi}{n}$ degree, j to denote the reflection about the principal axis and e to denote the identity. This definition leads to the following two conditions

(1) $r^n = e$
(2) $j^2 = e$

To communicate the rotation and the reflection, we add a third condition

(3) $r \circ j = j \circ r^{-1}$

We might want to observe an explicit example to get a better sense of what is going on here. When $n = 4$, the symmetries of square consist of rotations about the center and the reflections about the vertices and edges. The composition of the reflection and the rotation is same as the reflection along the diagonal.

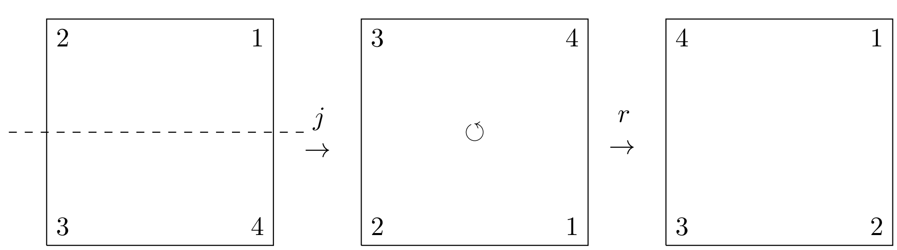

FIGURE 5. The Reflection along the Diagonal

1.6. **The Dihedral Group.** Let $D_n = \{e, r, r^2, \dots, r^{n-1}, j, r \circ j, \dots, r^{n-1} \circ j\}$ be the set of all symmetries of P_n. Then $(D_n, \circ)$ is a group. The order is $o(D_n) = 2n$. D_n can be generated as $\langle r, j \rangle$.

1.6.1. *The Rotation Group.* Let $C_n = \{e, r, r^2, \dots, r^{n-1}\} = \langle r \rangle$. Then C_n is a cyclic subgroup of D_n with order $o(C_n) = n$.

We can add more "fun" in the dihedral group. Instead of looking at the polygon P_n, let's grasp the essence and focus on the numbers.

Consider the set of functions $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$. The composition of function is associative. The identity function ε with $\varepsilon(k) = k$ is also contained in the set. We have to assign the inverses to make it a group.

1.7. **The Symmetric Group.** Let S_n be the set of *invertible* functions on $\{1, 2, \dots, n\}$. Then $(S_n, \circ)$ is a group. The order is $o(S_n) = n!$

1.7.1. *Cycle Notation of a Function.* To help us understand the symmetric group more explicitly, let's introduce the *cycle* notation. A cycle $\sigma = (a_1\ a_2\ \cdots\ a_n) \in S_n$ is a bijective function such that

$$\sigma(a_i) = a_{i+1 \mod n}.$$

Every function $f \in S_n$ can be represented as a product of disjoint cycles, as they are all invertible.

2. The Dihedral Group as a Subgroup of the Symmetric group

Let $(D_n, \circ)$ be the dihedral group of order $2n$. By labelling at the vertices of the geometrical representation of the regular convex n-gon P_n, we can construct an injection map from D_n to S_n. Let $\phi : D_n \to S_n$ be a group homomorphism such that

$$\phi(r) = (1\ 2\ \cdots\ n) \text{ and } \phi(j) = \prod_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} (i\ n+1-i).$$

Observe

$$\phi(r)^n = \varepsilon = \phi(j)^2$$

and for any $1 \le k \le n$,

$$\begin{aligned} \phi(j^{-1})\phi(r)\phi(j)(k) =& \phi(j)\phi(r)\phi(j)(k) \\ =& \phi(j)\phi(r)(n+1-k) \\ =& \phi(j)(n+2-k) && \mod n \\ =& (n+1) - (n+2-k) && \mod n \\ =& k-1. && \mod n \end{aligned}$$

This shows

$$\begin{aligned} & \phi(j^{-1})\phi(r)\phi(j) \\ =& (n\ n-1\ \cdots\ 1) \\ =& \phi(r)^{-1} \\ =& \phi(r^{-1}). \end{aligned}$$

since ϕ is a homomorphism. Then $\phi(r \circ j) = \phi(j \circ r^{-1})$.

Let's look at an example when $n = 4$.

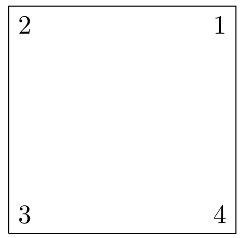

FIGURE 6. The Dihedral Group of Order Eight

By labelling the square, we construct the injection map $\phi : D_4 \to S_4$. The elements in D_4 are mapped to a subset of S_4,

$$\{\varepsilon, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2), (1\ 4)(2\ 3), (2\ 4), (1\ 2)(3\ 4), (1\ 3)\}.$$

This shows $D_4 \leqslant S_4$. This could be generalized to all $D_n \leqslant S_n$ as we've seen in class.

3. The Symmetries of Polyhedra as a Subgroup of the Symmetric Group

In last section, we've seen the dihedral group D_n as a subgroup of the symmetric group S_n. Specifically, we focused on the case when $n = 4$. Let's make it 3D.

Let S_P denote the symmetries of a regular polyhedra. Given there are 8 vertices of a cube, it's very likely that S_C would be a subgroup of S_8.

The idea is the same as the symmetries in $\mathbb{R}^3$ as still a rigid motion, hence it's bijective within S_C. We can generalize it to all convex polyhedra up to rigid motion. The non-convex polyhedra might also work, but the group associated might be trivial, ie, $S_P = \{e\}$.

Here's an interesting one. How about D_8 and S_C? D_8 has eight edges and an order 8 symmetry, the rotation r.

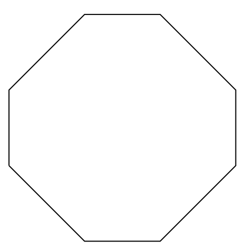

FIGURE 7. A Regular Convex Octagon

We can't really compare a 2D object and a 3D object at this time. To see how they are related, let's smash the cube on the plane.

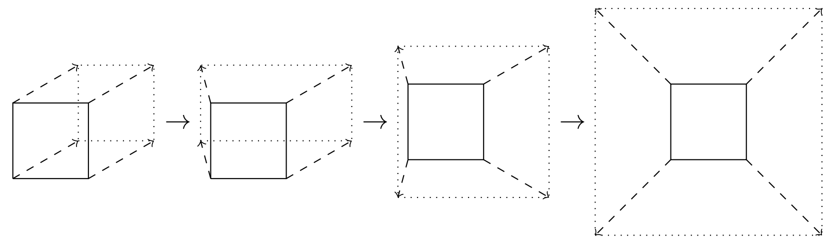

Figure 8. Continuously Deforming a Cube onto a Plane

There are four more edges in S_C than D_8, so there should be more symmetries on S_C than D_8.

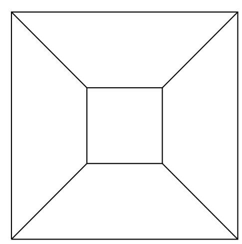

Figure 9. The Complete Planar Graph of the Symmetries of Cube

In the Introduction to Abstract Algebra course, we learnt that the order of the groups are $o(S_C) = 24$ and $o(D_8) = 16$. Since $\gcd(24, 16) = 8$, the Lagrange's Theorem tells us there should be no relationship between S_C and D_8.

However, if we consider the reflection along a principal plane, or the "oriented symmetry" of S_C , as paraphrased from my friend David of Western and quoted from an anonymous friend at Berkely, the order of S_C gets doubled to 48.

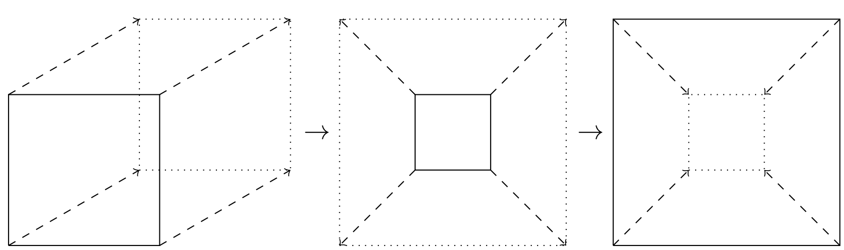

Figure 10. The Reflection along the Plane

This is one approach to tackle this problem. The plane symmetry could be represented by a 90° counterclockwise rotation γ of the planar graph of S_C.

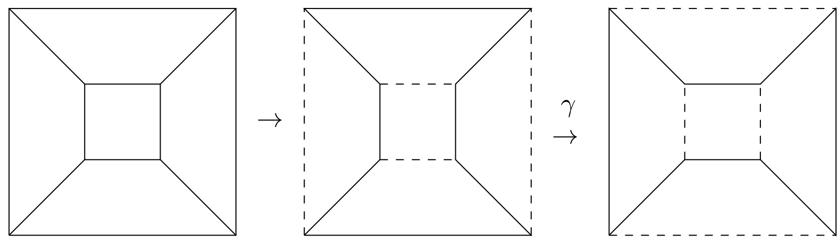

FIGURE 11. The Reflection along the Plane

I call these two shapes as the "sandglass" and the "butterfly"; they are the fundamental graphs of the symmetries of cube. The "sandglass" and the "butterfly" are graph isomorphic. The graph isomorphism is γ.

The other approach relies on "deforming" the non-convex octagon to a convex octagon. However, as suggested by the anonymous couple of Berkeley, this presentation defines an injection to S_4, or a bijection to D_4, as the $45°$ rotation destroys the symmetry. In other words, the group of symmetries of the "sandglass" is isomorphic to D_4.

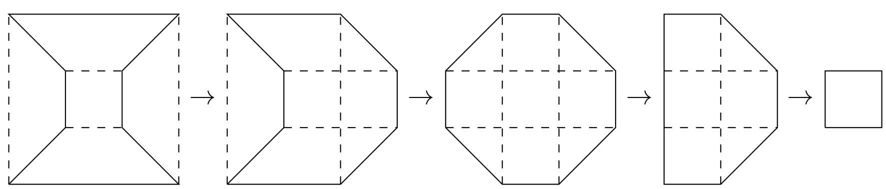

FIGURE 12. Continuously deforming the "butterfly" into a Square

More explicitly, let $G_C = (V_C, E_C)$ denote the complete planar graph of the cube. Let $\text{Aut}(G_C)$ denote the set of automorphisms on G_C. Then $\Phi : \text{Aut}(G_C) \to S_C$ defines a one-to-one map, or a *monomorphism*, from the symmetries of the cube to the symmetries of the graph. Moreover, this is a group monomorphism as

$$\Phi(\text{Aut}(G_C)) \simeq D_4 \leqslant S_C$$

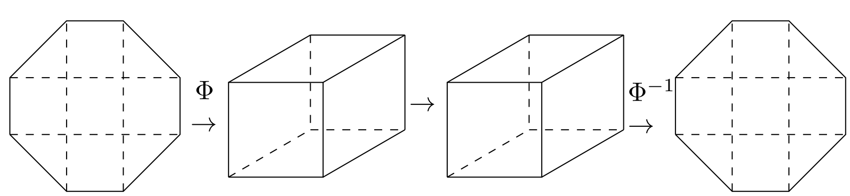

FIGURE 13. From the Symmetries of Graph to the Symmetries of Cube

However, this only tells us the *one-sided* story of the symmetires of the cube, as we are basically viewing the cube from its face, square. What's missing here?

4. Linear Algebra Takes in the Place

We would lose some information when smashing the cube onto the plane. For example, the angles between the edges lost there original symmetry. This will be a very interesting open question, so we'll save in the final section. Let's get back to $\mathbb{R}^3$ and make things precise again.

Consider a unit cube with center positioned at the origin. Aside from the position of vertices, edges and faces, the position of the *diagonals* also doesn't change. If we connect the vertices across the center of the cube, the symmetries of the cube would act as permutations on the four diagonals. By labelling the vertices we could change the cube into a "three dimensional" graph $G'_C = (V'_C, E'_C)$ where

$$V'_C = \{i : 1 \leq i \leq 8\}$$

and

$$E'_C = \{\{i, i+4\} : 1 \leq i \leq 4\}$$

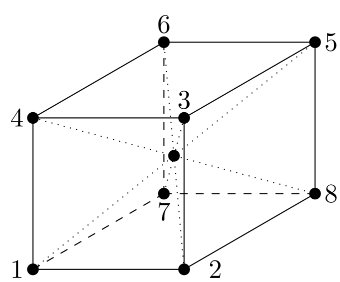

Figure 14. The Four Main Diagonals of the Square

Let r be the 90° rotation about the axis through the midpoints of a pair of opposite faces, r^2 be the 180° rotation about the axix through midpoints of a pair of opposite faces, d be the 120° rotation about the long diagonal, m be the 180° rotation about the axis through midpoints of a pair of opposite edges and e be the identity.

Let $e_i = \{i, i+4\} \in E'_C$. Let $\sigma = (e_i\ e_j)$ be a cycle notation which switches v_i and v_j. For each element is S_C, we have

Order	Elements in S_C	Cycle Notation
1	e	ε
2	m	$(e_i\ e_{(i \mod 4)+1})$
2	r^2	$(e_i\ e_j)(e_k\ e_l)$
3	d	$(e_i\ e_j\ e_k)$
4	r	$(e_i\ e_j\ e_k\ e_l)$

Figure 15. The Cycle Notation of Elements in S_C

Let $\phi : S_C \to S_4$ with $\phi(\sigma) = (i\ j)$. ϕ is a group isomorphism since $\phi(e) = \varepsilon$ and $\phi(\sigma\tau) = (i\ j)(k\ l)$ for $\sigma = (e_i\ e_j)$ and $\sigma = (e_k\ e_l)$. This shows $S_C \simeq S_4$.

Indeed, as agreed with $o(S_C) = 24 = o(S_4)$.

How about the "butterfly"? Let's establish cartesian coordinate in $\mathbb{R}^3$ and use matrices to clean up the dusts on the butterfly.

Let O be the center of the cube with side length 2 and embed the cube in $\mathbb{R}^3$. The matrix representation of the elements in S_C are

Elements in S_C	Matrix Representation
E as the identity transformation	$E = I_3$
M about the plane $x = y$	$M = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
R^2 about the x-axis	$R^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$
D about the vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$	$D = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
R about the x-axis	$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

FIGURE 16. The Matrix Representation of Elements in S_C

Notice the determinant of the matrices in FIGURE 16 is 1. Then the matrix representation of S_C forms a subgroup of $SL_3(\mathbb{Z})$, the special linear group of degree 3. Moreover, since the rotations and reflections are rigid motions, the matrix representation of S_C forms a subgroup of $O_3(\mathbb{Z})$, the orthogonal group of degree 3. Hence the matrix representation of S_C forms a subgroup of $SO_3(\mathbb{Z})$, the special orthogonal group of degree 3. Moreover, the group of matrix representations of S_C is the subgroup $SO_3(\mathbb{Z})$. Let $A \in SO_3(\mathbb{Z})$. Then

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

such that

$$\begin{cases} a^2 + d^2 + g^2 = 1 \\ b^2 + e^2 + h^2 = 1 \\ c^2 + f^2 + i^2 = 1 \end{cases} \quad \begin{cases} ab + de + gh = 0 \\ ac + df + gi = 0 \\ bc + ef + hi = 0 \end{cases} \quad aei + bfg + cdh - ceg - bdi - afh = 1$$

When $a = \pm 1$, $d = g = 0$ so $b = c = 0$. Then the conditions reduce to

$$\begin{cases} e^2 + h^2 = 1 \\ f^2 + i^2 = 1 \end{cases} \quad \begin{cases} ef + hi = 0 \end{cases} \quad aei - afh = 1$$

Then $ef = hi = 0$, otherwise the first system of equations fails. Suppose $e = 0$. Since e, h can't both be zero, then $i = 0$. Thus the third condition reduces to $-afh = 1$. If $a = 1$ then $h = \pm 1$ so $f = \mp 1$. So there are 4 choices when $e = 0$. Similiarly there are other 4 choices when $f = 0$. By symmetry in $a^2 + d^2 + g^2 = 1$, there are $3 \cdot 8 = 24$ elements in $SO_3(\mathbb{Z})$. This proves $S_C = SO_3(\mathbb{Z})$.

Let's add the plane reflection γ to S_C.

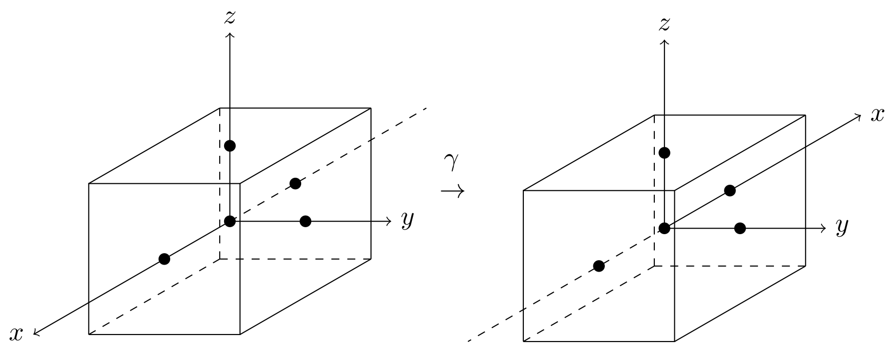

FIGURE 17. The Reflection about the yz-plane

The reflection along the yz-plane could be represented by matrix

$$\Gamma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The orientation of the matrix reverses as $\det(\Gamma) = -1$. The oriented group of matrix representations of S_C changes to $SO_3^{\pm}(\mathbb{Z}) = \{B \in O_3(\mathbb{Z}) : \det(B) = \pm 1\}$. Let $\gamma : SO_3(\mathbb{Z}) \to SO_3^{\pm}(\mathbb{Z})$ defined as $\gamma(B) = \Gamma B$. Then $\gamma(SO_3(\mathbb{Z})) = SO_3^{-}(\mathbb{Z})$, where $SO_3^{-}(\mathbb{Z}) = \{B \in O_3 : \det(B) = -1\}$. This shows

$$o(SO_3^{\pm}(\mathbb{Z})) = 2 \cdot o(SO_3(\mathbb{Z})) = 48$$

If we switch back the notation from matrices to the symmetric groups, we will see

$$S'_C \simeq S_4 \oplus \langle \gamma \rangle \simeq S_4 \oplus C_2$$

as $o(\gamma) = 2$.

This also presents an alternative interpretation for D_4 if we embed it in $\mathbb{R}^2$.

Since $C_4 = \langle r \rangle$ and $D_4 = \langle r, j \rangle$, the rotation $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has determinant $\det(R) = 1$. However, the reflection $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ has determinant $\det(J) = -1$.

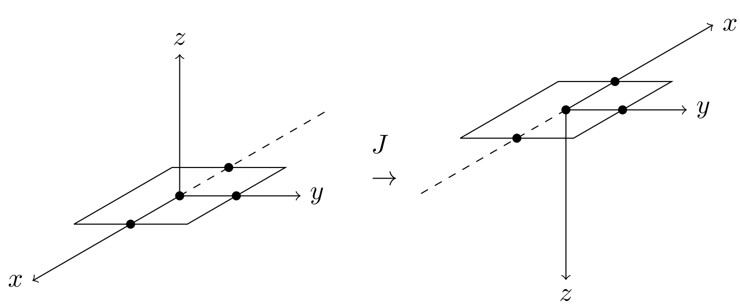

FIGURE 18. Reflection as an Oriented Rotation in $\mathbb{R}^3$

By the similar argument, the group of the matrix representations of C_4 is $SO_2(\mathbb{Z})$, and the group of the matrix representations of D_4 is the subgroup $SO_2^{\pm}(\mathbb{Z})$.

5. What Does the Complex Box Say

Let summarize what we've found so far about the symmetries of the cube S_C.

(1) In terms of symmetric groups, $S_C \simeq S_4$. For the matrices, $S_C \simeq SO_3(\mathbb{Z})$.
(2) If we assign an *orientation* on the cube, the oriented symmetries of the cube S'_C becomes $S_4 \oplus C_2$ and $SO_3^{\pm}(\mathbb{Z})$.
(3) We can establish an epimorphism from S_C to $\text{Aut}(G_C)$, where G_C is the complete planar graph of the cube.
(4) The highest order of a symmetry is 4.

These doesn't quite answer my last question as we still can't generate an element of order 8. Given the cube must be transformed under rigid motion, it would be impossible to generate an order 8 element. Consider these two graphs before we give up. If we trace a Eulerian path on the edges of the cube, we could potentially have a "symmetry" which has order 8.

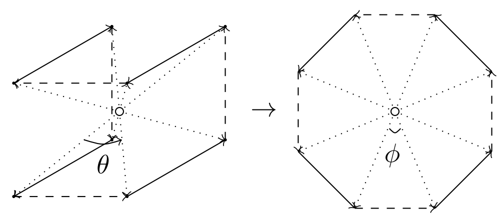

Figure 19. Hypothetical Epimorphism from S_C to C_8

However, we would lose some information when transform S_C into C_8. The original trace would follow a counterclockwise rotation θ such that $\tan\frac{\theta}{2} = \frac{1}{\sqrt{2}}$. So $\theta = 2\tan^{-1}\frac{\sqrt{2}}{2} \approx 70.529°$.

On the other hand, $\phi = \frac{360°}{8} = 45°$. We would lose approximately $25.529°$ in terms of the rotational information.

Can we make it up with a complex variable, as it contains two coordinates? We might want to construct a function that is real in one coordinate and *imaginary* in the others. The real component would dominate in both directions, and the imaginary component *always* winds back to its original position, so the mapping is still real. In other words, the sandglass would stand forever, but the butterfly fades away. We only take glimpse as she flutters in the mysterious fourth dimension.

This is still an open question. Some more traditional people believe this chapter would end with a solid period in Section 4. However, I still insist there are dreams and hopes lying in the hollowing complex plane...